Available online at www.sciencedirect.com
ScienceDirect
International Journal of Heat and Mass Transfer 50 (2007) 1823–1832
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Method of synthetic constraint, Fermat's principle and the constructal law in the fundamental principle of conductive heat transport

A.K. Pramanick, P.K. Das *

Department of Mechanical Engineering, Indian Institute of Technology, Kharagpur 721 302, India

Received 19 May 2006; received in revised form 29 September 2006
Available online 29 November 2006

Abstract

This article documents the method of synthetic constraint, a physical principle, to be applicable in the fundamental methodology of conductive heat flow, in replacement of calculus of variations and other optimal control theories. In particular, the optimum distribution of limited volume of insulating material on one side of a plane wall as well as cylindrically curved surface is obtained when the amount of insulating material is noninfluential to the imposed exponential temperature profile. The same physical theory is exercised for a generalized case of a stream suspended in an environment of different temperature and where the exponential wall temperature distribution is affected by the amount of insulation added. The result obtained conforms to those existing in open literature. Further from the physics of the problem it has been argued that a minimum exists for such class of problems of heat transfer from an insulated wall. Finally, it has been synthesized that Schmidt's criterion for the fin design, the tangent law of conductive heat transport and Fermat's principle in geometrical optics are but special stipulations of the method of synthetic constraint, which in turn is a corollary of constructal law. Thus the basis for analogies among physical theories is sought. The fundamental solution exhibits a category of equipartition principle.

Keywords: Constructal law; Control theory; Equipartition principle; Fermat's principle; Schmidt's criterion; Synthetic constraint; Thermal insulation; Variational calculus

1. Introduction

The problem of optimization is the very essence of reality [1,2]. It is well known that many physical theories naturally give rise to variational optimization principle from which the governing equations of the system can be deduced. The class of theories which do not yield a spontaneous variational formulation on account of nonlinearity or else can be modified to admit a variational form [3]. Inversely also it follows at once that the laws of physical theories when expressed as differential equations, the possibility of their reduction to a variational principle is evident from purely mathematical reasoning and does not depend on certain attributes intrinsic in the theory [4].

Despite these mathematical assertions, remarkably the classical thermodynamics [5,6] usually formulated is devoid of variational principles. However, it can be shown that as far as the implications for quasistatic transitions are concerned the second law can be formulated as a variational principle [7]. In classical mechanics it can be established that by means of Gauss's principle [8] all problems may be reduced to those pertaining to maxima and minima and hence possibly to a problem of variational calculus. Thus the variational technique as an optimization procedure has undergone tremendous upsurge both in science and engineering [9–13].

But the physicists and engineers often seem to disagree about the meaning of a variational principle [14]. For

* Corresponding author. Tel.: +91 3222 282916; fax: +91 3222 255303.
E-mail address: pkd@mech.iitkgp.ernet.in (P.K. Das).

0017-9310/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2006.10.003

Nomenclature

a	constant, Eq. (43)
ΔA	elemental surface area, Eq. (2a)
b	constant, Eq. (43)
Br_x	local Brun number, Eq. (45)
C	constant, Eq. (43)
C_{ra}	constant, Eq. (2b)
C_{sa}	constant, Eq. (2a)
C_{rv}	constant, Eq. (1b)
C_{sv}	constant, Eq. (1a)
C_{T}	constant, Eq. (31)
F	function of insulation thickness, Eq. (15a)
h	local convective heat transfer coefficient
h_i	heat transfer coefficient between stream and cylindrical wall, Eq. (32a)
h_0	heat transfer coefficient between insulation and ambient, Eq. (32a)
i	particular segment
k	local conductivity of insulating material
k_{i}	conductivity of insulating material, Eq. (32a)
k_{w}	conductivity of cylindrical wall, Eq. (32a)
L	wall length
m	number of segments
n	parameter determining wall temperature curvature
Nu_x	local Nusselt number, Eq. (43)
Δq	local heat transfer rate
ΔQ	form of local heat transfer rate, Eq. (3)
r	outer radius of cylindrical wall
t	insulation thickness
T	wall temperature
T_{f}	fluid stream temperature
T_L	wall temperature at $x = L$
T_0	wall temperature at $x = 0$
T_{w}	interfacial wall temperature, Eq. (40)
ΔT	local temperature gradient
t_{1l}	optimal thickness distribution, Eq. (21)
t_{2l}	optimal thickness distribution, Eq. (22)
t_{3l}	optimal thickness distribution, Eq. (23)
U	overall heat transfer coefficient, Eq. (32a)
Δv	form of elemental insulation volume, Eq. (3)
V	volume of insulating material
ΔV	elemental insulation volume
W	wall width
x	longitudinal coordinate

Greek symbols

δ	nondimensional parameter, Eq. (26b)
δ_{T}	thermal boundary layer thickness
Δ	dimensionless parameter, Eq. (27c)
ε	correction factor, Eq. (41)
λ	numerical and dimensional factor, Eqs. (1a) and (1b)
Λ_1	parametric group, Eq. (19)
Λ_2	parametric group, Eq. (27a)
Λ_3	parametric group, Eq. (29a)
Λ_4	parametric group, Eq. (30a)
μ	numerical and dimensional factor, Eqs. (2a) and (2b)
θ_1	angle of incidence, Eqs. (37b), (37c)
θ_2	angle of refraction, Eqs. (37b), (37c)
ψ	numerical and dimensional factor, Eq. (3)

Subscripts

opt	optimal, Eq. (35)
1	refers to wall
2	pertains to insulation

Superscript

–	averaged nondimensional quantity

physicists the fundamental element is generally the existence of a Lagrangian function through which the governing equations of the system are obtained by taking the functional derivatives. The main appeal of Lagrange function is its power of synthesis. The whole physics of the problem is expressed in terms of a single function. But the Lagrangian in our extended sense exists only for dissipative systems. On the other hand, for engineers the main point often seems to be the existence of a variational technique, as clearly indicated by the type of approximation methods [15] employed in engineering optimization which are largely independent of the existence of a Lagrange function.

Variational principle can also be formulated [16] outside the postulate of minimum entropy production [17] and the concept of local potential [17]. Quite apart from variational formulation a wide class of practical optimization problems can be expressed in the form of Pontryagin maximum principle [19]. It is reported that the attempts to solve these problems by the method of classical calculus of variations are not attractive [20].

An optimization procedure, such as variational method is usually carried out halfway that is the values of the parameters of a trial function are found for which a property of the system under consideration, such as the energy reaches its optimum value [21]. This has limited the progress of variational formulation and solution of related problems. Admittedly, if we do not succeed in solving a mathematical problem, it is often because we have failed to recognize the more general standpoint from which the problem before us appears as a single link in a chain of related problems. This way to find general methods is certainly the most practical and the surest, for he who seeks the method without having a definite problem in mind seeks in vain [22]. Thus the current research methodology emphasizes on the physical understanding of the problem in thermodynamic optimization of systems in particular [23]. Effort in this direction has been made to the design of conjugate conductive–convective coupled insulation system.

The present contribution explores the method of synthetic constraint [23], a physical principle, to the design of conductive insulation systems, which was recently analyzed by the formal method of calculus of variation [24]. Thermodynamic optimization of insulation system is also historically important and still remains an active research frontier in contemporary heat transfer research. It is historically important because the chronology of entropy generation minimization field [25] began with the design of insulation systems subject to finite-size constraint [26]. It is an active area of research since power plant and refrigeration unit can be regarded as thermal insulation system [27] while accepting the general definition that the thermal insulation is a system that prevents two surfaces of different temperatures from coming into direct thermal communication. From the physical perspective of the problem it has been demonstrated that for such a class of optimization problem a truly minimum exists. Finally it has been argued that there should be some basis for analogies among physical theories [28]. Being persuaded by such basis the relation among the method of synthetic constraint [23], Fermat principle [29,30] and constructal law [31,32] from which geometric forms [33] can be deduced out of a single physics principle [31,32] is sought.

2. The physical principle in conduction heat transfer

To engineer the nature is to understand her first. In this endeavor we seek continually a more general principle than the existing till an all-encompassing theory is established. The speculative way of seeking a new law is but to guess it first [34]. Recently proposed method of synthetic constraint [23] is an effort in this directive. In a nutshell this principle enunciates to identify the "conservation" of some physical quantities as a physical principle of thermodynamic optimization. Existence of such "isolines" is one of the most fundamental characteristics of extremality. Guided by this line of thought we proceed to identify the contributing competing mechanisms that constitute the locus of the physical process path describing the isoline.

To illustrate the rudimental feature of this principle we first consider a plane wall of length L and width W perpendicular to the plane of the paper as is shown in Fig. 1. The wall temperature variation $T(x)$ is only along the longitudinal direction x. The fundamental question corners around how to distribute a finite amount of insulating material either with constant or varying thickness $t(x)$ on the wall for minimum heat loss.

The insulated wall can be thought of being pieced into m equal or unequal length of sections. The more the local distribution of unit insulation material ΔV, the less is the local heat transfer rate Δq in general. On the other hand making

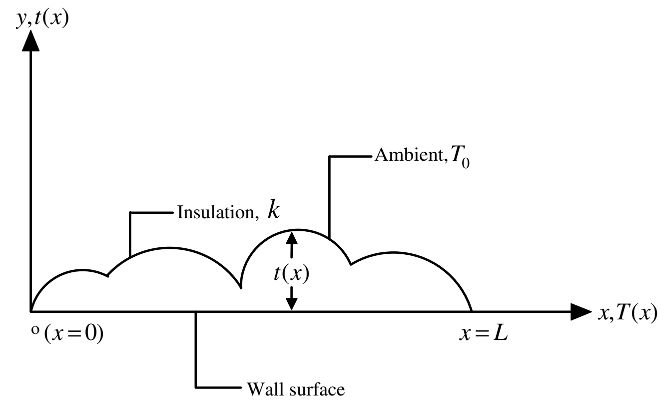

Fig. 1. Plane with arbitrary temperature profile and insulation distribution.

a particular segment of the wall more effective leads other parts of the wall to be less effective in insulation. Thus we identify heat transfer and insulation volume to be two competing physical factors (forces, motives) in insulation design. Hence, the legitimate postulate should be the uniform (equal) effectiveness of the insulation. This heuristic logic translates mathematically into

$$\Delta q_i + \lambda_i \Delta V = \Delta q + \lambda \Delta V = C_{sv} \tag{1a}$$

for $i = 1, 2, 3, \ldots, m$ and where C_{sv} is a constant. We drop the subscript i for equal segmentation. Here λ is a numerical and dimensional factor which makes the volume, a physical quantity, to be dimensionally homogeneous with another physical quantity heat. The far reaching consequences of this parameter in a greater perspective are to be realized [35]. The order of magnitude of the parameter λ is such that for which the problem of optimization is nontrivial. Hypothetically there may be some portion of the wall not covered with insulation at all, meaning $\lambda = 0$ as is in the leading and trailing edges of the wall. On the contrary, all insulation material can be applied on to a limited spot, leading $\lambda > 0$. Thus from the physical point of view the dimensional scale factor λ is bounded only in the domain $[0, \infty]$. To realize in another way the role played by λ, Eq. (1a) may be written in an alternative fashion when one of the constituents leads to a constant as

$$\frac{\Delta q_i}{\lambda_i \Delta V} = \frac{\Delta q}{\lambda \Delta V} = C_{rv} \tag{1b}$$

for $i = 1, 2, 3, \ldots, m$ and where C_{rv} is another constant. Notationally the subscript is dropped for equal segmentations as before. It can be seen that for $\lambda \to 0$, the constant on the right side of Eq. (1b) tends to a very high value meaning a very high rate of heat transfer as is also indicated by Eq. (1a) and thus not a desirable feature of modeling. On the other hand, for $\lambda \to \infty$ the constant on the right side of Eq. (1b) runs to a very low value implying a very low heat transfer as is also implied by Eq. (1a) and thus ensures a favorable modeling feature. But at the same time for the optimization problem to be nontrivial, the material volume cannot be unlimited or scarce posing a restriction to the upper and lower bound for the value of λ too.

Either Eq. (1a) or Eq. (1b) can be employed, as the case may be for the ease of computation or applicability, to obtain optimal profile of insulation in connection with minimum heat transfer from the wall with definite curvature and temperature profile.

If we consider insulating a line element instead of a plane wall, Eqs. (1a) and (1b) transforms respectively into

$$\Delta q_i + \mu_i \Delta A = \Delta q + \mu \Delta A = C_{sa} \tag{2a}$$

and

$$\frac{\Delta q_i}{\mu_i \Delta A} = \frac{\Delta q}{\mu \Delta A} = C_{ra} \tag{2b}$$

where the volume element (ΔV) is replaced by the surface area element (ΔA) and the dimensional role of μ has been changed to that of λ.

It is interesting to report that Eq. (2b) resembles Schmidt's idea [36] of optimum profile shape for cooling fin with minimum weight. At the same time it is to be noted that Schmidt's criterion were obtained upon a different heuristic logic. The intuitive logic of Schmidt was confirmed through rigorous variational formulation by Duffin [37]. Jany and Bejan [38] appeared at the conclusion that the idea of fin shape optimization has an important analog in the design of long ducts for fluid flow.

It is quite thought provoking that the problem for maximum heat transfer objective resembles to the challenge of insulation design for minimum heat transfer. It truly reflects the opposing action of two motive forces [39] in apparently two antagonistic arrangements. The physical factor which transcribes a problem of insulation into a question of fin is the curvature of the surface in consideration. For example critical insulation thickness [40] exists only in reality for the design of cylindrical and spherical layers, but not in the sizing of plane or nearly plane layers. Thus we repeat the symmetric appearance of a physical principle [41,42] with respect to its foundation in mathematical terms [28].

3. The physical basis for minimum heat transfer in insulation

The criteria for distinguishing between the maximum and minimum values of the functional have been investigated by many eminent mathematicians [43]. A rigorous mathematical discussion of the discriminating conditions may be found from the fundamental principle alone [44]. In our present endeavor we will however, provide a physical basis for the existence of the extremum. To be specific with the domain of application of this analysis we take the example of purely conductive insulation system.

From the physical perspective heat transfer and insulation volume are both nonnegative quantities. It is to be noted that we did not adopt here a control volume approach so as to regard heat transfer as positive or negative with respect to the system in a conventional manner. Again Eq. (1a) truly represents a competition between two opposing tendencies of the system. Further their constancy of summation leads to the fact that increment of one quantity drives to the decrement of the other in numerical estimate. These logics translate into the following mathematical prescriptions:

$$\Delta q = \Delta Q^2, \quad \Delta V = \Delta v^2 \quad \text{and} \quad \lambda = -\psi. \tag{3}$$

Thus Eq. (1a) transforms into

$$\Delta Q^2 - \psi \Delta v^2 = C_{sv}. \tag{4}$$

Since we are interested in global extremum, integrating upon Eq. (4) over the entire length of the plate we have

$$\int_0^L (\Delta Q^2 - \psi \Delta v^2)\, dx = C_{sv} L. \tag{5}$$

As indicated by the first example of the use of trigonometric series in the theory of heat [45] we adopt Fourier expansion [46] for the pattern of distribution of insulating material in primitive variables to be

$$\Delta v = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi}{L} x \tag{6}$$

where a_m's are some constants compatible with the convergence of the series. Rearranging Eq. (1a) in the following form

$$\Delta q = C_{sv} - \lambda \Delta V \tag{7}$$

and recognizing that heat transfer takes place in a normal direction to the plane under consideration [47] we find a compatible [48,49] Fourier series as

$$\Delta Q = \sum_{m=1}^{\infty} \frac{m\pi}{L} a_m \cos \frac{m\pi}{L} x. \tag{8}$$

Invoking Parseval's theorem [50] to the relations (6) and (8) we arrive respectively at

$$\int_0^L \Delta v^2\, dx = \frac{L}{2} \sum_{m=1}^{\infty} a_m^2 \tag{9}$$

and

$$\int_0^L \Delta Q^2\, dx = \frac{L}{2} \sum_{m=1}^{\infty} \frac{m^2\pi^2}{L^2} a_m^2. \tag{10}$$

The mathematical prescription for the applicability of Parseval's theorem is that

$$\Delta v(0) = \Delta v(L) = 0 \tag{11}$$

and ΔQ whose square is Lebesgue integrable [51] over the interval [0, L]. From the physics of the problem these criteria are quite recognizable. Thus incorporating Eqs. (9) and (10) in Eq. (5) we have

$$C_{sv} = \frac{1}{2} \sum_{m=1}^{\infty} \left(\frac{m^2\pi^2}{L^2} - \psi \right) a_m^2. \tag{12}$$

Noting that the left hand side is a finite positive quantity and hence there exists a minimum for the parameter ψ in the range

$$\psi \leqslant \frac{\pi^2}{L^2}. \tag{13}$$

It is to be remarked that the physical role [52] played by the parameter ψ here is very much different than λ in Eq. (1a). The physical argument presented above has an easy extension to the Sturm–Liouville theory [53]. Thus we conclude that a truly minimum exists for this problem of insulation design. Next we will calculate only the optimum profile for different geometries and temperature distributions. Once thus obtained optimum profile tallies with the established results the minimum heat transfer quantity follows at once.

4. Temperature distribution and heat transfer from an insulated wall

In many engineering applications [24] a nonlinear temperature variation $T(x)$ in the longitudinal direction x of the wall of finite length arises with definite curvature $\mathrm{d}^2T/\mathrm{d}^2x$. When the curvature of the wall temperature function is positive, temperature profile of the wall can be outlined as

$$\frac{T(x)-T_0}{T_L-T_0} = \frac{\exp\left(n\frac{x}{L}\right)-1}{\mathrm{e}^n-1} \tag{14a}$$

where T_0 and T_L are wall temperatures at $x=0$ and $x=L$ respectively and the nondimensional parameter n bears the same sign with the curvature of the wall temperature function. Here T_0 is also the ambient temperature. For the curvature of the temperature function of the wall to be negative, temperature distribution of the wall can be expressed algebraically as

$$\frac{T(x)-T_0}{T_L-T_0} = \frac{1-\exp\left(n\frac{x}{L}\right)}{1-\mathrm{e}^n}. \tag{14b}$$

In case of vanishingly small curvature of temperature function, passing to the limit $n \to 0$ either from Eq. (14a) or Eq. (14b) we obtain by applying L'Hospital's theorem a linear temperature distribution as

$$\frac{T(x)-T_0}{T_L-T_0} = \frac{x}{L}. \tag{14c}$$

On the other hand the mathematical advantage of the exponential representation of temperature is that it can be readily treated as differential equation [54]. Hence it may be possible to cast the whole exercise as a control problem of differential equation alone [55].

Recognizing the local temperature gradient $\Delta T = T(x) - T_0$ to be the cause of spontaneous heat transfer effect Δq in a coupled conductive–convective formulation [23] the expression for heat transfer stands as

$$\Delta q = \frac{\Delta T}{\frac{F[t(x)]}{k\Delta A} + \frac{1}{h\Delta A}} \tag{15a}$$

where ΔA is the elemental heat transferring area, k is the constant thermal conductivity of the insulating material, h is the local convective heat transfer coefficient, $t(x)$ is the local thickness of insulation and $F[t(x)]$ is the function of insulation thickness. Passing to the limit $h \to \infty$ in Eq. (15a) we arrive at

$$\lim_{h\to\infty} \Delta q = \lim_{h\to\infty} \frac{\Delta T}{\frac{F[t(x)]}{k\Delta A} + \frac{1}{h\Delta A}} = \frac{\Delta T}{\frac{F[t(x)]}{k\Delta A}}. \tag{15b}$$

Eq. (15a) is connected to Eq. (15b) in the same manner as the two-dimensional problem of heat transfer is related to the one-dimension when either of the dimensions is very great in comparison with the other.

In mathematical modeling of the problem we have both the choices: either to consider or not the effect of local insulation thickness on the driving force (ΔT) for the heat transfer.

5. Insulation on plane surface with static wall temperature condition

By static wall temperature condition we mean that the temperature distribution on the wall will not be affected by the amount of insulation mounted. We consider here a plane wall of length L and width W. The average insulation thickness $\bar{t}$ can be defined as

$$\bar{t} = \frac{V}{WL} = \frac{1}{L}\int_0^L t(x)\,\mathrm{d}x. \tag{16}$$

In Eq. (15b) we recognize for a plane wall that

$$\frac{F[t(x)]}{\Delta A} = \frac{t(x)}{W\,\mathrm{d}x}. \tag{17}$$

Now, employing the physical principle (1b) in Eq. (15b) along with Eq. (17) we directly obtain

$$t(x) = \left(\frac{\lambda}{k}\right)^{1/2}(\Delta T)^{1/2}. \tag{18}$$

For linear temperature distribution, invoking Eq. (14c) in Eq. (18) we arrive at

$$t(x) = \Lambda_1\left(\frac{x}{L}\right)^{1/2} \tag{19}$$

where Λ_1 is the shorthand for the constant $\left[\frac{\lambda}{k}(T_L-T_0)\right]^{1/2}$. Integrating Eq. (19) between 0 and L and employing Eq. (16) for the definition of average thickness we have

$$\Lambda_1 = \frac{3}{2}\bar{t}. \tag{20}$$

Optimal insulation profile is obtained by eliminating the constant Λ_1 between Eqs. (19) and (20) as

$$t_{1l}(x) = \frac{3}{2}\bar{t}\left(\frac{x}{L}\right)^{1/2}. \tag{21}$$

When the curvature of the wall temperature function is positive, employing Eq. (14a) in Eq. (18) and adopting similar procedure we get optimal insulation function as

$$t_{2l}(x) = \frac{n}{2}\bar{t}\frac{\sqrt{\exp\left(n\frac{x}{L}\right) - 1}}{\sqrt{e^n - 1} - \tan^{-1}\sqrt{e^n - 1}}. \quad (22)$$

For the curvature of the wall temperature profile to be negative recruiting Eq. (14b) in Eq. (18) similarly we obtain the optimal insulation profile as

$$t_{3l}(x) = \frac{n}{2}\bar{t}\frac{\sqrt{1 - \exp\left(n\frac{x}{L}\right)}}{\sqrt{1 - e^n} - \tanh^{-1}\sqrt{1 - e^n}}. \quad (23)$$

6. Insulation on cylindrical surface with static wall temperature condition

As stated before static wall temperature condition implies that the temperature of the wall is not a function of insulation volume. We consider now a cylinder of radius r and length L. Geometrically we mean a situation with the surface of revolution of the plane wall mounted with arbitrary insulation volume along with a translation in the vertical direction. Such a description bears an easy extension of the fundamental problem presented in Fig. 1. Then the fixed volume V of insulation is rendered by

$$V = \int_0^L \pi r^2\left\{\left[1 + \frac{t(x)}{r}\right]^2 - 1\right\}\mathrm{d}x. \quad (24a)$$

The relative thickness of insulation material is obtained in dimensionless form as

$$\overline{V} = \frac{V}{\pi r^2 L} = \frac{1}{L}\int_0^L \left\{\left[1 + \frac{t(x)}{r}\right]^2 - 1\right\}\mathrm{d}x. \quad (24b)$$

When the wall thickness is not negligible relative to the radius of the curvature of the wall surface, the problem must be analyzed by a method that takes the curvature into account. In Eq. (15b) we identify for a cylindrical wall [56]

$$\frac{F[t(x)]}{\Delta A} = \frac{\ln\left[1 + \frac{t(x)}{r}\right]}{2\pi\,\mathrm{d}x}. \quad (25)$$

Plugging Eq. (25) in lieu of Eq. (15b) along with Eq. (24a) into the physical principle (1a) we get the optimal insulation profile to comply with the following condition

$$\delta\ln\delta = \left(\frac{k}{\lambda r^2}\right)^{1/2}(\Delta T)^{1/2} \quad (26a)$$

where

$$\delta(x) = 1 + \frac{t(x)}{r}. \quad (26b)$$

Assuming a linear temperature distribution (14c) in Eq. (26a) we obtain

$$\delta\ln\delta = \Lambda_2\left(\frac{x}{L}\right)^{1/2} \quad (27a)$$

where Λ_2 is the notation for the parameter $\left[\frac{k}{\lambda r^2}(T_L - T_0)\right]^{1/2}$. The constant Λ_2 is determined from the definition (24b) as

$$\Lambda_2 = \left[\frac{2}{\overline{V}}\int_1^{\Delta}\delta(\delta^2 - 1)\ln\delta\ln(e\delta)\,\mathrm{d}\delta\right]^{1/2} \quad (27b)$$

where

$$\Delta = 1 + \frac{t_{\mathrm{opt}}(L)}{r}. \quad (27c)$$

Eliminating the constant Λ_2 between Eqs. (27a) and (27b) optimal insulation profile is obtained as

$$\delta\ln\delta = \left[\frac{2}{\overline{V}}\int_1^{\Delta}\delta(\delta^2 - 1)\ln\delta\ln(e\delta)\,\mathrm{d}\delta\right]^{1/2}\left(\frac{x}{L}\right)^{1/2}. \quad (28)$$

In the event of positive wall temperature curvature recruiting Eq. (14a) in Eq. (26a) we have

$$\delta\ln\delta = \Lambda_3\left[\exp\left(n\frac{x}{L}\right) - 1\right]^{1/2} \quad (29a)$$

where Λ_3 is the shorthand for the group $\left[\frac{k}{\lambda r^2}\frac{T_L - T_0}{e^n - 1}\right]^{1/2}$. The constant Λ_3 is implicitly determined using the definition (24b) as

$$\frac{2}{n}\int_1^{\Delta}\frac{\delta(\delta^2 - 1)\ln\delta\ln(e\delta)}{(\delta\ln\delta)^2 + \Lambda_3}\,\mathrm{d}\delta = \overline{V}. \quad (29b)$$

Eliminating the constant term Λ_3 from Eqs. (29a) and (29b) we obtain the required optimum insulation profile.

Similarly, for negative curvature of the wall temperature function employing Eq. (14b) in Eq. (26a) and exercising the same procedure we obtain the optimal profile of insulation as the eliminant of the parametric constant Λ_4 between the following equations

$$\delta\ln\delta = \Lambda_4\left[1 - \exp\left(n\frac{x}{L}\right)\right]^{1/2} \quad (30a)$$

and

$$\frac{2}{n}\int_1^{\Delta}\frac{\delta(\delta^2 - 1)\ln\delta\ln(e\delta)}{(\delta\ln\delta)^2 - \Lambda_4}\,\mathrm{d}\delta = \overline{V} \quad (30b)$$

where Λ_4 is the shorthand for the constant $\left[\frac{k}{\lambda r^2}\frac{T_L - T_0}{1 - e^n}\right]^{1/2}$.

7. Insulation on cylindrical surface with dynamic wall temperature condition

Unlike in Sections 5 and 6, we consider here a dynamic local temperature gradient situation for the wall. In other words, we do not neglect the effect of local insulation thickness on the local temperature distribution. Rather we impose the more realistic condition that the local temperature distribution is affected by the amount of insulation. Now, as a modeling feature we are at liberty to apply insulation in such a way that the local temperature potential remains piecewise constant i.e. $\Delta T \neq \Delta T(x)$. This makes in turn the local overall heat transfer coefficient U to be independent of longitudinal spatial position that is again $U \neq U(x)$. This idea of equipartitioned potential difference is directly borrowed from our recent work [57].

Let us consider a stream of fluid with local temperature distribution $T_f(x)$ passing through an insulated cylindrical tube whose outer surface is exposed to a constant environment temperature T_0 such that

$$\Delta T = T_f(x) - T_0 = C_T \tag{31}$$

where C_T is a constant.

The expression for overall heat transfer coefficient between the local bulk temperature of the stream $T_f(x)$ and the environment at T_0 can be readily obtained from any standard heat transfer textbook [56] as

$$\frac{1}{U2\pi r\,dx} = \frac{1}{h_0 2\pi[r+t(x)]\,dx} + \frac{\ln\left[1+\frac{t(x)}{r}\right]}{k_i 2\pi\,dx} + \frac{t_w}{k_w 2\pi r\,dx} + \frac{1}{h_i 2\pi r\,dx} \tag{32a}$$

where h_i and h_0 are the local convective heat transfer coefficients for the inner fluid and the outer fluid, k_i and k_w are the conductivities of the insulating material and cylinder wall respectively, r is the inner radius of the wall, t_w is the thickness of the wall. Recognizing the fact that h_i, $h_0 \to \infty$ and $\frac{t_w}{r} \to 0$, we pass on to these limits in Eq. (32a) to obtain

$$\frac{1}{U2\pi r\,dx} = \frac{\ln\left[1+\frac{t(x)}{r}\right]}{k_i 2\pi\,dx}. \tag{32b}$$

Putting Eq. (32b) in Eq. (25) along with Eq. (15b) into the physical principle (1a) we obtain

$$\left[\frac{\Delta T}{U2\pi r} + \lambda\pi r^2\left\{\left[1+\frac{t(x)}{r}\right]^2 - 1\right\}\right] dx = \text{constant}. \tag{33a}$$

By definition ΔT and U are constant and since dx can be arbitrarily small, the bracketed quantity on the right side vanishes identically i.e.

$$\frac{\Delta T}{U2\pi r} + \lambda\pi r^2\left\{\left[1+\frac{t(x)}{r}\right]^2 - 1\right\} = 0. \tag{33b}$$

Since $t(x)$ is the only variable on the left side, the physical solution of the equation leads to the fact that

$$t(x) = \text{constant}. \tag{34}$$

The constant of the right side of Eq. (34) is determined from Eq. (24b) as

$$t_{opt} = r\left[\left(1+\overline{V}\right)^{1/2} - 1\right]. \tag{35}$$

Eq. (35) is an important result and was obtained using the calculus of variations [24] and optimal control theory [59] as reported in literature as well as traditionally practiced by engineers.

8. Synthetic constraint, Fermat's principle and the constructal law

For two different materials of the wall and the insulating volume to be in perfect thermal contact, the interfacial boundary conditions demand that [60]

$$-k_1\left(\frac{\partial T_1}{\partial y}\right)_{0^+} = -k_2\left(\frac{\partial T_2}{\partial y}\right)_{0^-} \tag{36a}$$

and

$$T_1 = T_2 \tag{36b}$$

where the subscripts 1 and 2 refers to the general wall and the insulating material respectively. Eq. (36a) can be written as

$$\frac{\left(\frac{\partial T_1}{\partial y}\right)_{0^+}}{\left(\frac{\partial T_2}{\partial y}\right)_{0^-}} = \frac{k_2}{k_1} \tag{37a}$$

which readily admits the following form

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{k_2}{k_1} \tag{37b}$$

where θ_1 and θ_2 are the angles of incidence and refraction respectively. In turn for small angles Eq. (37b) can also be written as

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{k'_2}{k'_1} \tag{37c}$$

where k'_1 and k'_2 can be thought of as modified thermal conductivities. However, the approximate form of Eq. (37b) reads as

$$\frac{\sin\theta_1}{\sin\theta_2} \approx \frac{k_2}{k_1} \tag{37d}$$

for small angles of incidence and refraction. For constant thermal conductivities each of the form contained in Eq. (37a)–(37c) can be represented respectively as

$$\left(\frac{\partial T_1}{\partial y}\right)_{0^+} + \left(\frac{\partial T_2}{\partial y}\right)_{0^-} = \text{constant} \tag{38a}$$

$$\tan\theta_1 + \tan\theta_2 = \text{constant} \tag{38b}$$

and

$$\sin\theta_1 + \sin\theta_2 = \text{constant}. \tag{38c}$$

It is to be noted that the message contained in Eqs. (38a)–(38c) are but principally the one and the same: the very proposition of synthetic constraint. Further it is to be noted that Eq. (37b) is a consequence of tangent law in heat conduction [61] whereas Eq. (37c) is an outcome of Fermat's principle and also modelled through dynamic programming approach [62].

Comparing Eq. (37b) with (37d) we observe that there is a sacrifice of degree of accuracy. This criterion of accuracy is to be judged from the pertinent application in question. For example let us consider the more generalized situation of coupled conductive–convective heat transport mechanism [23]. Approximation of surface heat flux at the solid surface of the form

$$-k_1\left(\frac{\partial T_1}{\partial y}\right)_{0^+} \approx \frac{k_1(\Delta T_1)_{\bar{t}}}{\bar{t}} \tag{39}$$

is valid for a linear temperature distribution across the wall according to the following relation

$$T_1 = T_{\mathrm{w}}(x) + \frac{(\Delta T_1)_{\bar{t}}}{\bar{t}} \tag{40}$$

where the subscript w refers to the interfacial condition based upon average thickness $\bar{t}$ of the insulation volume. According to the theory of similarity [63] for a nonlinear temperature variation across the wall we may write

$$\left(\frac{\partial T_1}{\partial y}\right)_{0^+} \approx \varepsilon \frac{(\Delta T_1)_{\bar{t}}}{\bar{t}} \tag{41}$$

where ε is a correction factor for the distorted temperature profile. The slope on the right side of Eq. (41) is a single-valued function of $\frac{(\Delta T_1)_{\bar{t}}}{\bar{t}}$. Eqs. (36a) and (41) can be rearranged into the form

$$\frac{(\Delta T_1)_{\bar{t}}}{(\Delta T_2)_{\delta_{\mathrm{T}}}} = \varepsilon \frac{k_2}{k_1} \frac{t}{x} \frac{x}{\delta_{\mathrm{T}}} \tag{42}$$

where δ_{T} is the thermal boundary thickness of the medium. Approximate general local Nusselt number correlation can be expressed in the form

$$Nu_x = \frac{x}{\delta_{\mathrm{T}}} = CPr^a Re_x^b \tag{43}$$

where a, b and C are some constants. Thus the relative temperature drop term $(\Delta T_r)_{\bar{t}}$ contained in Eq. (42) is expressible as

$$(\Delta T_r)_{\bar{t}} = \frac{(\Delta T_1)_{\bar{t}}}{(\Delta T_2)_{\delta_{\mathrm{T}}}} = \varepsilon C f\left(\frac{k_2}{k_1} \frac{t}{x} Pr^a Re_x^b\right). \tag{44}$$

Clearly $(\Delta T_r)_{\bar{t}}$ is a single-valued function of the parametric group

$$Br_x = \frac{k_2}{k_1} \frac{t}{x} Pr^a Re_x^b \tag{45}$$

known as local Brun number. This local Brun number criterion [64] determines the degree of accuracy surrendered solving a conjugate problem as nonconjugate one.

In view of this engineering approximation either of Eqs. (37a)–(37c) or (37d) can be quantitatively treated in the form of unrestrictive synthetic constraint as

$$\theta_1 + \theta_2 = \text{constant}. \tag{46}$$

However, qualitatively ordinary optical rays obey Riemmanian geometry while thermal rays are described by Finslerian geometry [65]. In Eqs. (37b) and (37d) it is being revealed that between tangent law of heat conduction and Fermat's principle in optics there exists difference only in the degree of accuracy. Philosophically they are but the same: the unique optimization strategy of nature. Comparing Eqs. (37b) and (37c) it can be perceived that the tangent law of conductive heat transfer pertaining to a combination of mediums (k_1, k_2) is equivalent to Fermat's principle of optics to an altered combination of mediums (k_1', k_2'). Unlike point-to-point flow the demarcation between Fermat type flow and the constructal law is well established in the pertinent literature [57,66]. The Fermat type principle can be treated as a corollary of constructal law. It is also to be remarked that the method of synthetic constraint represents a category of equipartition [23,57,67,68] principle in some macroscopic domain with some finite time and length scale.

9. Conclusions

There has been mathematical studies towards the nonstandard methods in the calculus of variations [69]. However, the present study is under the proposition of a physical theory: the method of synthetic constraint [23]. It has been suggested in some authoritative treatises that in many problems where we only want a few values of the nonlinear partial differential equation, we can solve the associated variational problems instead [70]. Method of synthetic constraint is a justification on the physical basis in this direction.

Specifically the method of synthetic constraint has been exploited to a class of purely conductive system where some limited amount of insulating material is to be distributed over a plane wall or cylindrically curved surface with arbitrary temperature distributions for minimum heat transfer. The method is also extended to a more generalized situation of a stream suspended in an environment of different temperature and where the wall temperature distribution is affected by the amount of insulation added. The results obtained are in conformity with those reported in literature [24,59]. The equivalence of the result obtained in applying the variational principle for a prescribed temperature history to that obtained for a prescribed heat flux is well established in pertinent literature [71].

From the physics of such class of extremum problems it has been argued that a truly minimum exists. However, the quantification of minimum heat transfer has not been reported here. Once the optimum profile of insulation is obtained the minimum heat transfer quantity follows readily from the routine procedure and is available in the literature [24]. Since any distribution pattern of insulting material can be represented by a Fourier series, it has been insinuated that such class of conductive minimum heat transfer problems pertain to a category of Sturm–Liouville system.

Finally, from a summation from of synthetic constraint, a ratio form is derived when one of the constituent competing mechanisms turns out to be a constant. Thus the ratio form of synthetic constraint is more restrictive than its summation counterpart. It turns out to be a mathematical fact that when the ratio form is valid the summation form is spontaneously granted but not the vice versa. In view of this argument Schmidt's criterion for the fin design, the tangent law of conductive heat transport and the Fermat's law of geometrical optics obey the principle of synthetic constraint which in turn is a corollary of constructal law. Thus the basis for analogies among some physical theories is sought. The fundamental feature of this optimization is but a category of macroscopic organization with a class of equipartition principle [23,57,67,68].

References

[1] V.M. Tikhomirov, Stories about Maxima and Minima, Mathematical World, vol. 1, American Mathematical Society, 1990, pp. 3–8.
[2] R. Courant, H. Robbins, What is Mathematics? Oxford University Press, Oxford, 1941, pp. 329–397.
[3] E. Tonti, A systematic approach to the variational formulation in physics and engineering, in: Autumn course on Variational Methods in Analysis and Mathematical Physics, International Centre for Theoretical Physics, Trieste, 20 October–11 December, 1981.
[4] W. Yourgrau, S. Mandelstam, Variational Principles in Dynamics and Quantum Theory, Dover, New York, 1979, p. 175.
[5] A. Bejan, Advanced Engineering Thermodynamics, Wiley, New York, 1997, pp. 29–38.
[6] C. Truesdell, The Tragicomedy of Classical Thermodynamics, International Centre for Mechanical Sciences, Udine, Courses and Lectures, No. 70, Springer-Verlag, New York, 1983.
[7] H.A. Buchdahl, A variational principle in classical thermodynamics, Am. J. Phys. 55 (1987) 81–83.
[8] H. Hancock, The Theory of Maxima and Minima, Dover, New York, 1960, pp. 150–151.
[9] B.A. Finlayson, L.E. Scriven, On the search for variational principles, Int. J. Heat Mass Transfer 10 (1967) 799–821.
[10] M.A. Biot, Variational Principles in Heat Transfer, Oxford University Press, Oxford, 1970.
[11] H.H. Goldstine, A History of the Calculus of Variations from the 17th through 19th Century, Springer-Verlag, New York, 1980.
[12] I. Todhunter, A History of the Calculus of Variations During the Nineteenth Century, Dover, New York, 2005.
[13] S. Sieniutycz, Progress in variational formulations of macroscopic processes, in: S. Sieniutycz, H. Farkas (Eds.), Variational and Extremum Principles in Macroscopic Systems, Elsevier, London, 2004.
[14] I. Prigogine, Remarks on variational principles, in: R.J. Donnelly, R. Herman, I. Prigogine (Eds.), Non-Equilibrium Thermodynamics, Variational Techniques and Stability, Chicago University Press, Chicago, 1965.
[15] L.V. Kantrovich, V.I. Krylov, Approximations of Higher Analysis, Wiley, New York, 1958.
[16] R.S. Schechter, Variational principles for continuum systems, in: R.J. Donnelly, R. Herman, I. Prigogine (Eds.), Non-Equilibrium Thermodynamics, Variational Techniques and Stability, Chicago University Press, Chicago, 1965.
[17] I. Prigogine, Evolution criteria, variational properties and fluctuations, in: R.J. Donnelly, R. Herman, I. Prigogine (Eds.), Non-Equilibrium Thermodynamics, Variational Techniques and Stability, Chicago University Press, Chicago, 1965.
[19] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, Pergamon Press, Oxford, 1964.
[20] A.A. Fel'dbaum, On the question of synthesizing optimum automatic control systems (in Russian), Transactions of the Second All Union Conference on Automatic Control Theory, vol. 2, USSR Academy of Science, 1955.
[21] M.J. ten Hoor, The variational method—why stop half way? Am. J. Phys. 62 (1994) 166–168.
[22] D. Hilbert, Mathematical Problems, Arch. Math. Phys. 3 (1901) 1, 44–63, 213–237.
[23] A.K. Pramanick, P.K. Das, Heuristics as an alternative to variational calculus for optimization of a class of thermal insulation systems, Int. J. Heat Mass Transfer 48 (2005) 1851–1857.
[24] A. Bejan, How to distribute a finite amount of insulation on a wall with nonuniform temperature, Int. J. Heat Mass Transfer 36 (1993) 49–56.
[25] A. Bejan, Second law analysis in heat transfer and thermal design, Adv. Heat Transfer 15 (1982) 1–58.
[26] A. Bejan, Entropy Generation Minimization, CRC Press, Boca Raton, New York, 1996, pp. 155–169.
[27] A. Bejan, A general variational principle for thermal insulation system design, Int. J. Heat Mass Transfer 22 (1979) 219–228.
[28] E. Tonti, The reason for analogies between physical theories, Appl. Math. Modell. 1 (1976) 37–50.
[29] W.A. Newcomb, Generalized Fermat principle, Am. J. Phys. 51 (1983) 338–340.
[30] D. Censor, Fermat's principle and real space time rays in absorbing media, J. Phys. A: Math. Gen. 10 (1977) 1781–1790.
[31] A. Bejan, Shape and Structure, from Engineering to Nature, Cambridge University Press, Cambridge, 2000.
[32] R.N. Rosa, A.H. Reis, A.F. Miguel (Eds.), Bejan's Constructal Theory of Shape and Structure, Évora Geophysics Center, University of Évora, Évora, Portugal, 2004.
[33] D.S. Lemons, Perfect Form, Princeton University Press, Princeton, New Jersey, 1997, pp. ix–xi.
[34] R. Feynman, The Character of Physical Law, MIT Press, Massachusetts, 1967, pp. 149–173.
[35] H.S. Leff, What if entropy were dimensionless? Am. J. Phys. 67 (1999) 1114–1122.
[36] E. Schmidt, Die Wärmeübertragung durch Rippen, Z. Ver. Dt. Ing. 70 (1926) 885–889, 947–951.
[37] R.J. Duffin, A variational problem relating to cooling fins, J. Math. Mech. 8 (1959) 47–56.
[38] P. Jany, A. Bejan, Ernst Schmidt's approach to fin optimization: an extension to fins with variable conductivity and the design of ducts for fluid flow, Int. J. Heat Mass Transfer 31 (1988) 1635–1644.
[39] R. Clausius, On the motive power of heat and on the laws which can be deduced from it for the theory of heat (W.F. Magie, Trans.), in: E. Mendoza (Ed.), Reflections on the Motive Power of Fire by Sadi Carnot and Other Papers on the Second Law of Thermodynamics by É. Clapeyron and R. Clausius, Dover, New York, 1960.
[40] A. Bejan, Heat Transfer, Wiley, New York, 1993, pp. 42–44.
[41] R. Feynman, The Character of Physical Law, MIT Press, Massachusetts, 1967, pp. 84–107.
[42] B.C. van Fraassen, Laws and Symmetry, Oxford University Press, Oxford, 1989.
[43] I. Todhunter, A History of the Calculus of Variations During the Nineteenth Century, Dover, New York, 2005, pp. 243–253.
[44] E.P. Culverwell, On the discrimination of maxima and minima solutions in the calculus of variations, Phil. Trans. Roy. Soc. Lond. A 178 (1887) 95–129.
[45] J. Fourier, The Analytical Theory of Heat (A. Freeman, Trans.), Dover, New York, 1955, pp. 137–144.
[46] H.S. Carslaw, Introduction to the Theory of Fourier Series and Integrals, Dover, New York, 1930, pp. 323–328.
[47] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, Oxford, 1959, pp. 6–8.
[48] E.T. Whittaker, G.N. Watson, A Course on Modern Analysis, Cambridge University Press, Cambridge, 1996, pp. 224–225.
[49] G.P. Tolstov, Fourier Series (R.A. Silverman, Trans.), Dover, New York, 1976, pp. 12, 60.
[50] H.S. Carslaw, Introduction to the Theory of Fourier Series and Integrals, Dover, New York, 1930, pp. 284–288.
[51] H.S. Carslaw, Introduction to the Theory of Fourier Series and Integrals, Dover, New York, 1930, pp. 329–361.
[52] P.W. Bridgman, Tolman's principle of similitude, Phys. Rev. 8 (1916) 423–431.
[53] R. Bellman, Methods of Nonlinear Analysis, vol. I, Academic Press, New York, 1970, pp. 304–330.
[54] R. Courant, Differential and Integral Calculus, vol. I (E.J. McShane, Trans.), Wiley, New York, 1967, pp. 178–179.
[55] R. Bellman, Introduction to the Mathematical Theory of Control Process, vol. I, Academic Press, New York, 1970, pp. 33–34.
[56] A. Bejan, Heat Transfer, Wiley, New York, 1993, p. 40.
[57] A.K. Pramanick, P.K. Das, A note on constructal theory of organization in nature, Int. J. Heat Mass Transfer 48 (2005) 1974–1981.

[59] M. Kalyon, A.Z. Sahin, Application of optimal control theory in pipe insulation, Numer. Heat Transfer Part A 41 (2002) 391–402.
[60] M.N. Özişik, Heat Conduction, Wiley, New York, 1993, pp. 17–20.
[61] A. Tan, L.R. Holland, Tangent law of refraction for heat conduction through an interface and underlying variational principle, Am. J. Phys. 58 (1990) 988–991.
[62] S. Sieniutycz, Dynamic programming approach to a Fermat type principle for heat flow, Int. J. Heat Mass Transfer 43 (2000) 3453–3468.
[63] L.I. Sedov, Similarity and Dimensional Methods in Mechanics, Mir Publishers, Moscow, 1982.
[64] A.V. Luikov, Conjugated heat transfer problems, Int. J. Heat Mass Transfer 3 (1961) 293–303.
[65] H. Janyszek, R. Mrugala, Riemannian and Finslerian geometry and fluctuations of thermodynamic systems, in: S. Sieniutycz, P. Salamon (Eds.), Nonequlibrium Theory and Extremum Principles, Taylor & Francis, New York, 1990.
[66] A. Bejan, Constructal comment on a Fermat-type principle for heat flow, Int. J. Heat Mass Transfer 46 (2003) 1885–1886.
[67] A. Bejan, D. Tondeur, Equipartition, optimal allocation and the constructal approach to predicting organization in nature, Rev. Gen. Therm. 37 (1998) 165–180.
[68] A. De Vos, B. Desoete, Equipartition principle in finite-time thermodynamics, J. Non-Equilib. Thermodyn. 25 (2000) 1–13.
[69] C. Tuckey, Nonstandard Methods in the Calculus of Variations, Pitman Research Notes in Mathematics Series, vol. 297, Longman Scientific & Technical, Essex, 1993.
[70] R. Bellman, Selective Computation, World Scientific, Philadelphia, 1985, p. 38.
[71] T.J. Lardner, Biot's variational principle in heat conduction, AIAA J. 1 (1963) 196–206.